JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **225,** 91–113 (1998)
ARTICLE NO. AY986008

A Random Fixed Point Theorem and the Random Graph Transformation

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Submitted by Jack K. Hale

Received August 5, 1997

In the first part of this article we formulate a random fixed point theorem for random dynamical systems where a contraction condition is formulated as an expectation of a particular expression. In the following we are going to formulate a random graph transformation. This transformation defines a random dynamical system. Under certain conditions this random dynamical system has a random fixed point which can be found by the mentioned random fixed point theorem. In one application we show that random dynamical systems given by particular differential equations have a random unstable invariant manifold. © 1998 Academic Press

1. INTRODUCTION

The *graph transformation* is an important tool in the theory of dynamical systems. For instance, this transformation can be used to prove the existence of *invariant manifolds*, see Babin and Vishik [2, Chap. 5], Marsden and McCracken [13, Chap. 1], Hirsch, Pugh, and Shub [11]. Roughly speaking, we call such a manifold *stable* if the states on this manifold tend exponentially fast to a steady state of the dynamical system as time t tends to infinity. The same is true for the *unstable* manifold as time t tends to $-\infty$, see Hale [10], Babin and Vishik [2].

A generalization of deterministic or autonomous systems are the *random dynamical systems*. Important examples of these dynamical systems are differential equations or difference equations with time-dependent and

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0022-247X/98 $25.00

random coefficients. This time dependence is given by the operators of a *flow* which leave a probability measure invariant. In this sense we assume that the coefficients of the equations that define random dynamical systems are stationary processes. For the exact definition of a random dynamical system see the following text.

In the first part we formulate a random fixed point theorem for random dynamical systems. Theorems which state the existence of random fixed points are formulated for instance in Bharuchia-Reid [3], Engl [8], Hong-Kun Xu [12], Papageorgiou [14], and Tzu-Chu Lin [17]. We deal with a different kind of random fixed points. Our fixed point theorem allows us to find unique stationary solutions of random dynamical systems which are stable. It is a random version of the well-known Banach fixed point theorem which are applied to a set of random variables fulfilling certain growth conditions w.r.t. the measure preserving flow called *temperedness*. It is important to note that the contraction condition is only defined in the average. This random fixed point theorem is a generalization of a fixed point principle in Schmalfuss [15] in which the existence of strong stationary solutions for stochastic partial differential equations was proved assuming the existence of positively invariant sets.

In the second part of the article we deal with a graph transformation for random dynamical systems. We prove that this transformation defines a new, a *lifted* random dynamical system defined on a function space.

By the observation that the random graph transformation is in itself also a random dynamical system we are in a position to use the random fixed point theorem to prove the existence of *stationary solutions*.

In the third part of the article we consider an example demonstrating how to use the random graph transformation and the random fixed point theorem to find the unstable manifold of a random dynamical systems. This random dynamical system is given by a semicoupled random differential equation. Stable and unstable manifolds for random dynamical systems were investigated by Wanner [18]. To prove the existence of these invariant manifolds he uses particular deterministic fixed point theorems on subsets of functions satisfying exponential growth conditions. Another kind of invariant manifold for random dynamical systems was studied in Boxler [4] and Dahlke [7].

However, we are going to find random invariant manifolds by another technique based on the random graph transformation. In our approach we formulate the assumption for the Lipschitz constants in a more general *averaged* sense which differs from Wanner's and Boxler's considerations where the assumptions concerning the Lipschitz constant of the nonlinear part are deterministic (uniformly bounded). In this approach we only study random differential equations which are decoupled in one direction. For more general examples we refer to Schmalfuss [16].

2. A RANDOM FIXED POINT THEOREM

Here we give the definition of random dynamical systems for which we later establish a fixed point theorem. A model describing random perturbations is a *metric dynamical system*. A metric dynamical system is given by $\theta := (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{T}})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\theta_t\}_{t \in \mathbb{T}}$ is a *flow* of $\mathbb{P}$-preserving transformations for time $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$,

$$\begin{aligned} \theta_{t+\tau} &= \theta_t \circ \theta_\tau, \quad \text{for all } t, \tau \in \mathbb{T}, \\ \theta_0 &= \mathrm{id}. \end{aligned}$$

We use the notation $\theta_t \theta_\tau = \theta_t \circ \theta_\tau$. If $\mathbb{T} = \mathbb{R}$, then we also assume the $(\mathscr{B}_{\mathbb{R}} \otimes \mathscr{F}, \mathscr{F})$-measurability of

$$\mathbb{R} \times \Omega \ni (t, \omega) \mapsto \theta_t \omega \in \Omega.$$

In both cases, we assume the $\{\theta_t\}_{t \in \mathbb{T}}$-ergodicity of $\mathbb{P}$.

Let us note that a metric dynamical system is a general model for a (*random*) *noise*. Suppose that G is a complete metric space. A *cocycle* is a mapping,

$$\phi : \mathbb{T} \times \Omega \times G \to G,$$

such that for any $t, \tau \in \mathbb{T}$, $\omega \in \Omega$, $x \in G$,

$$\begin{aligned} \phi(t + \tau, \omega, x) &= \phi(t, \theta_\tau \omega, \cdot) \circ \phi(\tau, \omega, x), \\ \phi(0, \omega, x) &= x. \end{aligned} \tag{1}$$

Later on we need a more general definition of a cocycle. Suppose we have a family of nonempty sets $\{G(\omega)\}_{\omega \in \Omega}$. We assume that we have a mapping ϕ,

$$(t, (\omega, x)) \in \mathbb{T} \times \bigcup_{\omega \in \Omega} (\{\omega\} \times G(\omega)) \to \phi(t, \omega, x) \in G(\theta_t \omega), \tag{2}$$

such that the property (1) is fulfilled. A cocycle ϕ which is $(\mathscr{B}_{\mathbb{T}} \otimes \mathscr{F} \otimes \mathscr{B}_G, \mathscr{B}_G)$-measurable or in the sense of (2) measurable with respect to the appropriate trace σ-algebra is called *measurable*. The pair (θ, ϕ) is called *random dynamical system*. Many publications contain a definition of a cocycle which implies the measurability assumption automatically. However, for our construction later on it would be too strong to assume *this* measurability. If the mapping $x \mapsto \phi(t, \omega, x)$ is continuous for any $t \in \mathbb{T}$, $\omega \in \Omega$, then we have a *continuous* cocycle.

More generally, we could assume that the cocycle is defined only for nonnegative time $\mathbb{T}^+$. In the following we formulate a fixed point theorem for random variables. This theorem states the existence of random fixed points or stationary solutions $g^*(\omega)$ for random dynamical systems such that

$$\phi(t, \omega, g^*(\omega)) = g^*(\theta_t \omega), \tag{3}$$

for $t \in \mathbb{T}$ and $\omega \in \Omega'$ where $\Omega' \subset \Omega$ is a θ-invariant set of full $\mathbb{P}$-measure.

Let us introduce a particular class of ω-dependent mappings. We call a mapping f on Ω with values in $\mathbb{R}^+$ *tempered* if

$$\lim_{t \to \pm\infty} \frac{\log^+ f(\theta_t \omega)}{|t|} = 0, \tag{4}$$

for $\omega \in \Omega$. This condition is equivalent to the subexponential growth of $t \mapsto f(\theta_t \omega)$,

$$\lim_{t \to \pm\infty} e^{-c|t|} f(\theta_t \omega) = 0, \quad \text{for any } c > 0,\ \omega \in \Omega. \tag{5}$$

Suppose f is a random variable, then (4) has only one alternative property: on a set of full measure we have

$$\limsup_{t \to \pm\infty} \frac{\log^+ f(\theta_t \omega)}{|t|} = +\infty.$$

In particular, we have the following sufficient condition for temperedness of a random variable f on a θ-invariant set $\Omega' \subset \Omega$ of full measure,

$$\begin{aligned} \mathbb{E} \log^+ f < \infty, &\quad \text{if } \mathbb{T} = \mathbb{Z}, \\ \mathbb{E} \sup_{t \in [0,1]} \log^+ f(\theta_t \omega) < \infty, &\quad \text{if } \mathbb{T} = \mathbb{R}, \end{aligned} \tag{6}$$

which is a simple consequence of the Birkhoff ergodic theorem. On the other hand, if f is majorized by a tempered mapping, then f is tempered itself. Moreover, the sum or the product of finitely many tempered mappings is also tempered.

Remark 2.1. Sometimes when we have a metric dynamical system with $\mathbb{T} = \mathbb{R}$ we are only interested in temperedness w.r.t. the restricted metric dynamical system with flow $\{\theta_i\}_{i \in \mathbb{Z}}$.

In what follows we are going to formulate a fixed point theorem for random variables generalizing Schmalfuss [15], which is a random analogous of the Banach fixed point theorem. In particular, we show the existence of a unique random fixed point (or stationary solution) for a random dynamical system. A contraction condition is only satisfied on the average.

Before we state the theorem we note the following simple fact based on the Birkhoff ergodic theorem. Let $k \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and let θ be a metric dynamical system with time $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. Then there exists a $\{\theta_i\}_{i \in \mathbb{Z}}$ invariant set $\overline{\Omega}$ of full measure such that

$$\lim_{i \to \pm\infty} \frac{1}{|i|} \sum_{j=0}^{i \mp 1} k\big(\theta_{-j}\,\omega\big) = \mathbb{E}k, \quad \text{for } \omega \in \overline{\Omega}. \tag{7}$$

THEOREM 2.2. *Let $(G(\omega), d_\omega)$ be a family of metric spaces which are subspaces of a metric space (G, d) such that the restriction of d on $G(\omega)$ is equivalent to d_ω and let ϕ be a random dynamical system with a cocycle in the sense of (2) on the metric dynamical system θ over a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where $\mathbb{P}$ is ergodic w.r.t. θ. The cocycle ϕ which is supposed to be continuous is defined on nonnegative time $\mathbb{T}^+$. Let k be an integrable random variable such that $\mathbb{E}k =: K < 0$. Let $\Omega' \subset \overline{\Omega}$ (defined in (7)) be a θ-invariant set of full measure and let $\mathscr{G}_{\Omega'}$ be a nonempty set of random variables $\omega \mapsto g(\omega) \in G(\omega)$ having the following properties*:

- Mapping into: *We suppose*

$$\omega \mapsto \phi\big(t, \theta_{-t}\,\omega, g(\theta_{-t}\,\omega)\big) \in \mathscr{G}_{\Omega'}, \tag{8}$$

for $g \in \mathscr{G}_{\Omega'}$, $t \in \mathbb{T}^+$.

- "Completeness": *If the sequence $(\phi(t, \theta_{-t}\,\omega, g(\theta_{-t}\,\omega)))$, $g \in \mathscr{G}_{\Omega'}$ for $t \to \infty$ is a Cauchy sequence for any $\omega \in \Omega'$, then this limit has to be in $\mathscr{G}_{\Omega'}$.*
- Contraction: *We assume that for $\omega \in \Omega'$,*

$$\sup_{\gamma_1 \neq \gamma_2 |(\omega)} \log \frac{d_{\theta_1\omega}(\phi(1, \omega, \gamma_1), \phi(1, \omega, \gamma_2))}{d_\omega(\gamma_1, \gamma_2)} \leq k(\omega), \tag{9}$$

if $G(\omega)$ contains more than one element.

- Temperedness: *In case of discrete time we assume that*

$$\omega \mapsto d_\omega(g(\omega), \bar{g}(\omega)), \qquad g, \bar{g} \in \mathscr{G}_{\Omega'}$$

is tempered. In case of continuous time this mapping and

$$\omega \mapsto \sup_{s\in[0,1]} d_\omega\big(\phi(s,\theta_{-s}\omega, g(\theta_{-s}\omega)), g(\omega)\big) \tag{10}$$

are tempered in the sense of Remark 2.1.

Then there exists a fixed point $g^* \in \mathscr{G}_{\Omega'}$ for the restriction of ϕ and θ on Ω' fulfilling (3)*. This fixed point is unique. In addition, we have the following convergence with exponential rate,*

$$\lim_{t\to\infty} d_\omega\big(\phi(t,\theta_{-t}\omega, g(\theta_{-t}\omega)), g^*(\omega)\big) = 0, \qquad \omega \in \Omega', \tag{11}$$

$$\lim_{t\to\infty} d_{\theta_t\omega}\big(\phi(t,\omega, g(\omega)), g^*(\theta_t\omega)\big) = 0, \quad a.s. \tag{12}$$

for any $g \in \mathscr{G}_{\Omega'}$.

Proof. We prove the continuous time case $\mathbb{T}^+ = \mathbb{R}^+$. The discrete time case is a simple version of this case.

(i) For any $\varepsilon > 0$, $g_1, g_2 \in \mathscr{G}_{\Omega'}$ and $\omega \in \Omega'$ there exists an $i_0(\omega, \varepsilon, g_1, g_2)$ such that for $i > i_0$,

$$\begin{aligned}
&d_\omega\big(\phi(i,\theta_{-i}\omega, g_1(\theta_{-i}\omega)), \phi(i,\theta_{-i}\omega, g_2(\theta_{-i}\omega))\big)\\
&\quad\le d_\omega\big(\phi(1,\theta_{-1}\omega, \phi(i-1,\theta_{-i}\omega, g_1(\theta_{-i}\omega))),\\
&\qquad\qquad \phi(1,\theta_{-1}\omega, \phi(i-1,\theta_{-i}\omega, g_2(\theta_{-i}\omega)))\big)\\
&\quad\le e^{k(\theta_{-1}\omega)} d_{\theta_{-1}\omega}\big(\phi(i-1,\theta_{-i}\omega, g_1(\theta_{-i}\omega)),\\
&\qquad\qquad\qquad \phi(i-1,\theta_{-i}\omega, g_2(\theta_{-i}\omega))\big)\\
&\quad\le \exp\left(\sum_{j=1}^{i} k(\theta_{-j}\omega)\right) d_{\theta_{-i}\omega}\big(g_1(\theta_{-i}\omega), g_2(\theta_{-i}\omega)\big) < e^{(1/2)Ki} < \varepsilon,
\end{aligned}$$

by the temperedness of $\omega \mapsto d_\omega(g_1(\omega), g_2(\omega))$.

(ii) It follows by the estimate in (i) and by the cocycle property,

$$\begin{aligned}
&d_\omega\big(\phi(i,\theta_{-i}\omega, g(\theta_{-i}\omega)), \phi(i+1,\theta_{-i-1}\omega, g(\theta_{-i-1}\omega))\big)\\
&\quad\le \exp\left(\sum_{j=1}^{i} k(\theta_{-j}\omega)\right) d_{\theta_{-i}\omega}\big(g(\theta_{-i}\omega), \phi(1,\theta_{-i-1}\omega, g(\theta_{-i-1}\omega))\big).
\end{aligned}$$

(iii) For $0 \le t \le t_1$ one has by (i) and (ii),

$$d_\omega\Big(\phi\big(t, \theta_{-t}\omega, g(\theta_{-t}\omega)\big), \phi\big(t_1, \theta_{-t_1}\omega, g\big(\theta_{-t_1}\omega\big)\big)\Big)$$

$$\le \exp\left(\sum_{j=1}^{[t]} k\big(\theta_{-j}\omega\big)\right)$$

$$\times\, d_{\theta_{-[t]}\omega}\Big(\phi\big(t-[t], \theta_{-t+[t]}\theta_{-[t]}\omega, g\big(\theta_{-t+[t]}\theta_{-[t]}\omega\big)\big),$$

$$\phi\big(t_1-[t], \theta_{-t_1+[t]}\theta_{-[t]}\omega, g\big(\theta_{-t_1+[t]}\theta_{-[t]}\omega\big)\big)\Big)$$

$$\le \exp\left(\sum_{j=1}^{[t]} k\big(\theta_{-j}\omega\big)\right)$$

$$\times\Bigg(d_{\theta_{-[t]}\omega}\Big(\phi\big(t-[t], \theta_{-t+[t]}\theta_{-[t]}\omega, g\big(\theta_{-t+[t]}\theta_{-[t]}\omega\big)\big), g\big(\theta_{-[t]}\omega\big)\Big)$$

$$+\sum_{j=0}^{[t_1]-[t]-1} d_{\theta_{-[t]}\omega}\Big(\phi\big(j, \theta_{-[t]-j}\omega, g\big(\theta_{-[t]-j}\omega\big)\big),$$

$$\phi\big(j+1, \theta_{-[t]-j-1}\omega, g\big(\theta_{-[t]-j-1}\omega\big)\big)\Big)$$

$$+d_{\theta_{-[t]}\omega}\Big(\phi\big([t_1]-[t], \theta_{-[t_1]}\omega, g\big(\theta_{-[t_1]}\omega\big)\big),$$

$$\phi\big(t_1-[t], \theta_{-t_1+[t]}\theta_{-[t]}\omega, g\big(\theta_{-t_1+[t]}\theta_{-[t]}\omega\big)\big)\Big)\Bigg).$$

The last expression can be written as

$$d_{\theta_{-[t]}\omega}\Big(\phi\big([t_1]-[t], \theta_{-[t_1]}\omega, g\big(\theta_{-[t_1]}\omega\big)\big),$$

$$\phi\Big([t_1]-[t], \theta_{-[t_1]}\omega,$$

$$\phi\big(t_1-[t_1], \theta_{-t_1+[t_1]}\theta_{-[t_1]}\omega, g\big(\theta_{-t_1+[t_1]}\theta_{-[t_1]}\omega\big)\big)\Big)\Big). \tag{13}$$

Using the abbreviations $[t] = i$ and $l(\omega) := 2\sup_{s\in[0,1]} d_\omega(\phi(s, \theta_{-s}\omega, g(\theta_{-s}\omega)), g(\omega)))$ one can show from (ii) and (13) the following estimate which is true for any $t_1 \ge t$,

$$d_\omega\Big(\phi\big(t, \theta_{-t}\omega, g(\theta_{-t}\omega)\big), \phi\big(t_1, \theta_{-t_1}\omega, g\big(\theta_{-t_1}\omega\big)\big)\Big)$$

$$\le \exp\left(\sum_{j=1}^{i} k\big(\theta_{-j}\omega\big)\right)\sum_{m=0}^{\infty}\left(\exp\left(\sum_{j=1}^{m} k\big(\theta_{-i-j}\omega\big)\right) l\big(\theta_{-i-m}\omega\big)\right). \tag{14}$$

The right-hand side of the last formula is finite for $\omega \in \Omega'$, $t \geq 0$ by the temperedness of l (see (10)) and

$$\sum_{j=1}^{m} k\big(\theta_{-i-j}\omega\big) \approx Km \to -\infty, \quad \text{for } m \to \infty, \text{ for any } i \in \mathbb{Z}^+.$$

We now show that the right hand side of (14) tends to zero for $i \to \infty$. Because the first factor on the right-hand side of (14) has an exponential decay we are done if we can show that

$$\sum_{m=0}^{\infty} \exp\left(\sum_{j=1}^{m} k\big(\theta_{-i-j}\omega\big)\right) l(\theta_{-i-m}\omega) \tag{15}$$

has a subexponential growth with respect to i (see (5)), for $\omega \in \Omega'$. We can rewrite and we can estimate the interior terms of (15) replacing ω by $\theta_{-i}\omega$ by

$$\exp\left(\sum_{j=1}^{m+i} \big(k\big(\theta_{-j}\omega\big) - K\big) - \sum_{j=1}^{i} \big(k\big(\theta_{-j}\omega\big) - K\big) \right.$$
$$\left. + mK + \log^+ l(\theta_{-i-m}\omega) \right). \tag{16}$$

For any $\varepsilon > 0$, $\omega \in \Omega'$ we have $i_0(\varepsilon, \omega)$ such that for any $i \geq i_0(\varepsilon, \omega)$,

$$\log^+ l(\theta_{-i}\omega) \leq \varepsilon i, \qquad \left| \sum_{j=1}^{i} \big(k\big(\theta_{-j}\omega\big) - K\big) \right| \leq \varepsilon i.$$

The first estimate follows from the temperedness of l and the second estimate follows from (7). In particular, we have for $i \geq i_0$,

$$\log^+ l(\theta_{-i-m}\omega) \leq \varepsilon(i+m), \qquad \left| \sum_{j=1}^{i+m} \big(k\big(\theta_{-j}\omega\big) - K\big) \right| \leq \varepsilon(i+m),$$

where ε is independent of $m \in \mathbb{Z}^+$. Let c be an arbitrary positive number. We choose an ε such that

$$0 < \varepsilon < -\frac{1}{4}\max\left(-\frac{c}{2}, K\right).$$

Therefore,

$$e^{-(c/2)i}\sum_{m=0}^{\infty}\left(\exp\left(\sum_{j=1}^{m}k(\theta_{-i-j}\omega)\right)l(\theta_{-i-m}\omega)\right)\le\sum_{m=0}^{\infty}e^{-m\varepsilon}<\infty,$$

for $i>i_0$ and $\omega\in\Omega'$, hence,

$$\lim_{i\to\infty}e^{-ci}\left(\sum_{m=0}^{\infty}\exp\left(\sum_{j=1}^{m}k(\theta_{-i-j}\omega)\right)l(\theta_{-i-m}\omega)\right)=0, \tag{17}$$

which shows that (15) is tempered.

Summarizing, we found that $(\phi(t,\theta_{-t}\omega,g(\theta_{-t}\omega)))$ is a Cauchy sequence for $\omega\in\Omega'$, $g\in\mathscr{G}_{\Omega'}$. The limit of this sequence is denoted by $g^*(\omega)$. By assumption we know that $g^*\in\mathscr{G}_{\Omega'}$.

(iv) On account of the continuity of ϕ the mapping g^* fulfills (3),

$$\begin{aligned}\phi(\tau,\omega,g^*(\omega))&=\phi\Big(\tau,\omega,\lim_{t\to\infty}\phi(t,\theta_{-t}\omega,g(\theta_{-t}\omega))\Big)\\&=\lim_{t\to\infty}\phi\big(t+\tau,\theta_{-t-\tau}\theta_\tau\omega,g(\theta_{-t-\tau}\theta_\tau\omega)\big)=g^*(\theta_\tau\omega),\end{aligned}$$

for any $\tau\ge 0$, $\omega\in\Omega'$.

(v) Assuming there exist two stationary solutions $g_1^*,g_2^*\in\mathscr{G}_{\Omega'}$. Then by (i) it follows that $g_1^*(\omega)=g_2^*(\omega)$, $\omega\in\Omega'$.

(vi) The convergence (11) follows from the stated Cauchy sequence property. For $t_1\to\infty$ we can replace in (14) $\phi(t_1,\theta_{-t_1}\omega,g(\theta_{-t_1}\omega))$ by $g^*(\omega)$ where the estimate in (14) remains true. Formula (17) shows that the convergence in (11) is exponentially fast. In addition, by Flandoli and Langa [9, Section 4] it follows the *forward* convergence of (12) with exponential rate almost surely. ∎

Remark 2.3. (i) For the discrete time case the invariance assumptions for $\mathscr{G}_{\Omega'}$ include that $d_\omega(\phi(1,\theta_{-1}\omega,g(\theta_{-1}\omega)),g(\omega))$ is tempered.

(ii) We can replace the assumption $k\in L^1$ by $k^+\in L^1$ and $\mathbb{E}k^-=\infty$. Indeed, in the proof of Theorem 2.2 we can replace k by $\max(k,-n)$, where $n\in\mathbb{N}$ is sufficiently large so that $\mathbb{E}\max(k,-n)<0$.

(iii) Assume the cocycle ϕ is defined for any $t\in\mathbb{T}$. Then the invariance assertion for g^* is also satisfied for any $t\in\mathbb{T}$.

(iv) An analysis of the proof shows that we only need the measurability of the cocycle and of the elements of $\mathscr{G}_{\Omega'}$ for the convergence property (12) and for the measurability of the fixed point g^*. However, without measurability of the cocycle and if $\mathscr{G}_{\Omega'}$ is only a particular set of ω-dependent mappings we can also find a fixed point which is not measurable, in general, such that (3) is fulfilled. Sometimes we assume measurability of the cocycle and the elements of $\mathscr{G}_{\Omega'}$ in a weaker form. This weaker measurability transfers to a weaker measurability of the fixed point g^*.

(v) In the formulation of the main theorem of this section we introduced metrics depending on ω. Therefore it is possible to formulate this fixed point theorem in the sense of *random norms*, see Arnold [1] or Wanner [18]. These norms can be used to describe the growth in particular directions.

(vi) Suppose g^* is generated by a $g \in \mathscr{G}_{\Omega'}$ as in (iii) of the proof of Theorem 2.2. Then one can show similarly to this proof that $\omega \mapsto d_\omega(g(\omega), g^*(\omega))$ is tempered in the sense of Remark 2.1. Thus, by the triangle inequality, $\omega \mapsto d_\omega(g(\omega), g^*(\omega))$ is also tempered for any $g \in \mathscr{G}_{\Omega'}$.

3. THE RANDOM GRAPH TRANSFORMATION

In this section we establish the concept random unstable manifold. In addition, we are going to show that the graph transformation for a random dynamical system also defines a cocycle where the state space is a function space. We assert that a random fixed point (or stationary solution) of this random dynamical system forms the graph of an unstable manifold.

Let $\theta := (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system, and consider a cocycle S on $\mathbb{R}^d$ with time set $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$ having the fixed point 0. (θ, S) is a random dynamical system over the state space $\mathbb{R}^d$.

We consider measurable multifunctions $M(\omega)$ which are invariant in the sense that

$$S(t, \omega, M(\omega)) \subset M(\theta_t \omega). \tag{18}$$

In particular, we consider sets $M(\omega)$ such that for $t \in \mathbb{R}$, $\omega \in \Omega$ for $x \in M(\omega)$ the states $S(t, \omega, x)$ tend exponentially fast to 0 for $t \to -\infty$. We call this set *random* unstable invariant set denoted by $M^+(\omega)$. From this decay condition and from the fact that the cocycle S is defined for any $t \in \mathbb{T}$ it can also be shown that one can replace in (18) $\subset$ by $=$, see Arnold [1]. In the same manner we could study random *stable* invariant sets $M^-(\omega)$, given by states that tend exponentially fast to zero for $t \to \infty$. But these considerations are essentially the same as for unstable sets. So we do not repeat this case.

We now suppose that S is a cocycle consisting of Lipschitz continuous mappings $S(t, \omega, \cdot)$ which are differentiable at $x = 0$, so we are able to determine the linearization of $S(t, \omega, \cdot)$ at $x = 0$ denoted by $S'(t, \omega, 0)$. For discrete time we assume that this linear operator satisfies

$$\mathbb{E}\left(\log^+\|S'(1, \omega, 0)\| + \log^+\|S'(1, \omega, 0)^{-1}\|\right) < \infty. \tag{19}$$

Similarly, for a random dynamical system with continuous time we assume that

$$\mathbb{E}\left(\sup_{t\in[0,1]} \log^+\|S'(t, \omega, 0)\| + \sup_{t\in[0,1]} \log^+\|S'(t, \omega, 0)^{-1}\|\right) < \infty.$$

By these assumptions we can apply the multiplicative ergodic theorem, see Arnold [1] or Wanner [18], which provides the existence of *Lyapunov exponents* $\lambda_1 > \lambda_2 > \cdots \lambda_p$, $p \le d$. By the assumption of ergodicity of the metric dynamical system these Lyapunov exponents are (almost surely) independent of ω. In particular, one can find a θ-invariant set of full measure such that for any ω in this set the assertions of the multiplicative ergodic theorem are fulfilled. Without loss of generality this set is denoted by Ω. From now on we assume that the fixed point 0 of the random dynamical system is *hyperbolic* which means that

$$\lambda_1 > \lambda_2 > \cdots \lambda_r > 0 > \lambda_{r+1} > \cdots > \lambda_p,$$

for some r satisfying $1 \le r \le p - 1$.

The multiplicative ergodic theorem also gives the existence of invariant unstable, stable linear subspaces $E^+(\omega), E^-(\omega)$ so that

$$\begin{aligned}
E^+(\omega) \cap E^-(\omega) &= \{0\},\\
E^+(\omega) \oplus E^-(\omega) &= \mathbb{R}^d,\\
S'(t, \omega, 0)E^+(\omega) &= E^+(\theta_t\omega),\\
S'(t, \omega, 0)E^-(\omega) &= E^-(\theta_t\omega).
\end{aligned}$$

The linear space $E^+(\omega)$ is defined to be the direct sum of the *Oseledets spaces* related to Lyapunov exponents bigger than 0. Similarly, we can construct the space $E^-(\omega)$. There also exist measurable projections $\pi_\omega^\pm$ from $\mathbb{R}^d$ onto $E^\pm(\omega)$.

The goal of this article is to prove the existence of a random unstable manifold $M^+(\omega)$ which is given as a *perturbation* of the unstable linear space $E^+(\omega)$. Unstable manifolds satisfy particular regularity conditions.

In the following we demonstrate our method to find Lipschitz-continuous manifolds. However, this method also enables us to find manifolds satisfying stronger regularity conditions. We start with some definitions of function spaces in which we seek for unstable manifolds. Let $C_0^{0,1}$ be the space of global Lipschitz continuous functions u on $\mathbb{R}^d$ with values in $\mathbb{R}^d$ such that $u(0) = 0$ where the norm is given by

$$\|u\|_L = \sup_{x_1 \neq x_2 \in \mathbb{R}^d} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|}.$$

In addition, we consider a second Banach space C_0^G of continuous functions u with $u(0) = 0$ such that the norm,

$$\|u\|_G := \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|u(x)|}{|x|} < \infty.$$

The space $C_0^{0,1}$ is continuously embedded in C_0^G. Using this function space we can define

Definition 3.1. Suppose there exists a mapping $g^*(\omega) \in C_0^G(E^+(\omega), E^-(\omega), E^-(\omega))$.[1] In addition, we suppose that for fixed $x \in \mathbb{R}^d$ the mapping $\Omega \ni \omega \mapsto g^*(\omega, x) \in \mathbb{R}^d$ is measurable, where $g^*(\omega, x)$ is the evaluation of the continuous function $g^*(\omega)$ at x. If the random unstable set is given by

$$M^+(\omega) = \{x = x^+ + g^*(\omega, x^+) : x^+ \in E^+(\omega)\}, \tag{20}$$

then $\{M^+(\omega)\}_{\omega \in \Omega}$ is called the random unstable invariant manifold of the random dynamical system (θ, S) at 0.

We mention that the measurability of the multifunction defined in (20) follows from continuity of $g^*(\omega)$:

Lemma 3.2. *Assume* $\{M^+(\omega)\}_{\omega \in \Omega}$ *is given by* (20). *Then this multifunction is measurable.*

Proof. $g^*(\omega)$ is a continuous function, hence $M^+(\omega)$ is closed. By Castaing and Valadier, [5, Lemma III.14], the mappings,

$$(\omega, x) \mapsto \pi_\omega^+ x, \qquad (\omega, x) \mapsto g^*(\omega, x)$$

[1] We identify $C_0^G(E^+(\omega), E^-(\omega))$ with the subspace of $C_0^G(\mathbb{R}^d, \mathbb{R}^d)$ such that $g^*(x) = \pi_\omega^- g^*(\pi_\omega^+ x)$.

are (jointly) measurable. Hence the mapping,

$$(\omega, x) \mapsto y - \pi_\omega^+ x - g^*(\omega, x) \tag{21}$$

is *measurable* for any $y \in \mathbb{R}^d$. To prove that $\{M^+(\omega)\}_{\omega \in \Omega}$ is measurable we have to show that

$$\inf_{x \in \mathbb{R}^d} |y - \pi_\omega^+ x - g^*(\omega, x)|$$

is measurable for any $y \in \mathbb{R}^d$, see Castaing and Valadier [5, Theorem III.9], which follows immediately from the continuity in x of (21). ∎

Remark 3.3. Similarly, we can define and we can prove the measurability of other kinds of manifolds which are invariant in the sense of (18).

In the following we are going to construct a cocycle on C_0^G. First we have to note some properties of this space.

LEMMA 3.4. *For any $r \geq 0$ the set,*

$$\overline{B}_{C_0^{0,1}}(0, r) = \left\{u \in C_0^{0,1} : \|u\|_L \leq r\right\}$$

is closed in C_0^G.

Proof. Let $(u_n) \in \overline{B}_{C_0^{0,1}}(0, r)$ be a C_0^G-converging sequence with limit u_0. Thus u_n tends to u_0 uniformly on any compact subset of $\mathbb{R}^d$. Hence we have for any $x_1 \neq x_2 \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{|u_n(x_1) - u_n(x_2)|}{|x_1 - x_2|} = \frac{|u_0(x_1) - u_0(x_2)|}{|x_1 - x_2|} \leq r,$$

such that $u_0 \in B_{C_0^{0,1}}(0, r)$. ∎

Let $G(\omega)$, $\omega \in \Omega$ be a subspace of functions in C_0^G. We suppose that these functions have values in $E^-(\omega)$. Similar to the deterministic graph transformation (see Babin and Vishik [2, Chap. 5] and Marsden and McCracken [13, Section 1]) we suppose that for $\gamma \in G(\theta_{-t}\omega)$,

$$E^+(\theta_{-t}\omega) \ni x^+ \mapsto S^+\left(t, \theta_{-t}\omega, x^+ + \gamma(x^+)\right) = y^+ \in E^+(\omega)$$

defines a bijection from $E^+(\theta_{-t}\omega)$ onto $E^+(\omega)$ for any $t \geq 0$, $\omega \in \Omega$, $\gamma \in G(\theta_{-t}\omega)$ where $S^{\pm}(t, \omega, \cdot) = \pi_{\theta_t\omega}^{\pm} S(t, \omega, \cdot)$. The inverse mapping is denoted by $T(t, \omega, \gamma)(y^+)$. We define the following mapping,

$$\phi(t, \omega, \gamma)(y^+) = S^-\left(t, \omega, T(t, \theta_t\omega, \gamma)(y^+) + \gamma\left(T(t, \theta_t\omega, \gamma)(y^+)\right)\right). \tag{22}$$

$\phi(t, \omega, \cdot)$ is only defined on $G(\omega)$ for $\omega \in \Omega$. The images are assumed to be contained in $G(\theta_t \omega)$. This mapping represents the *random graph transformation*. The *deterministic* graph transformation transforms a graph of a mapping into another graph (see Babin and Vishik [2, p. 224f]). To find an invariant manifold one has to look for a fixed point of this transformation. Similarly, in the random case we transform a graph by a random mapping. But it would not make sense to look for (nontrivial) fixed points of this mapping. Indeed, in contrast to the deterministic case the random graph transformation $\phi(t, \omega, \cdot)$ transforms a graph from $G(\omega)$ into a graph of another fiber $G(\theta_t \omega)$. To find the random analogous of a fixed point (a stationary solution) one needs a particular structure, which is a cocycle.

THEOREM 3.5. *Assume* $\{G(\omega)\}_{\omega \in \Omega}$ *is a multifunction such that for the mapping defined in* (22),

$$\phi(t, \omega, G(\omega)) \subset G(\theta_t \omega).$$

Then (θ, ϕ) *defines a cocycle in the sense of* (1) *and* (2).

Proof. For $t \geq 0$, $\omega \in \Omega$, $\gamma \in G(\omega)$ we set

$$\mu = \phi(t, \omega, \gamma). \tag{23}$$

First we show that

$$T(t+\tau, \theta_{t+\tau}\omega, \gamma)(z^+) = T(t, \theta_t \omega, \gamma)(\cdot) \circ T(\tau, \theta_{t+\tau}\omega, \mu)(z^+),$$
$$z^+ \in E^+(\theta_{t+\tau}\omega).$$

Let $x^+ \in E^+(\omega)$. Then we have

$$z^+ = S^+(t+\tau, \omega, x^+ + \gamma(x^+)) \quad \Leftrightarrow \quad x^+ = T(t+\tau, \theta_{t+\tau}\omega, \gamma)(z^+).$$

On the other hand, we have by $y^+ = S^+(t, \omega, x^+ + \gamma(x^+))$ the relation $T(t, \theta_t \omega, \gamma)(y^+) = x^+$. On account of (23) we have

$$z^+ = S^+\big(\tau, \theta_t \omega, S^+(t, \omega, x^+ + \gamma(x^+)) + \mu(S^+(t, \omega, x^+ + \gamma(x^+)))\big).$$

From the last equation we find

$$T(\tau, \theta_{t+\tau}\omega, \mu)(z^+) = y^+ = S^+(t, \omega, x^+ + \gamma(x^+)),$$

which gives the asserted proposition.

We now check that ϕ is a cocycle on $\{G(\omega)\}_{\omega \in \Omega}$. Equation (23) gives

$$\begin{aligned} \mu(y^+) &:= S^-\big(t, \omega, T(t, \theta_t \omega)(y^+) + \gamma\big(T(t, \theta_t \omega, \gamma)(y^+)\big)\big) \\ &= \phi(t, \omega, \gamma)(y^+), \qquad y^+ \in E^+(\theta_t \omega). \end{aligned}$$

For any $z^+ \in E(\theta_{t+\tau}\omega)$ we have

$$\begin{aligned}
\phi(\tau, \theta_t\omega, \mu)(z^+) &= S^-\big(\tau, \theta_t\omega, T(\tau, \theta_{t+\tau}\omega, \mu)(z^+) \\
&\qquad + \mu\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big)\big) \\
&= S^-\big(\tau, \theta_t\omega, S\big(t, \omega, T(t, \theta_t\omega, \gamma) \\
&\qquad \circ \big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big) \\
&\qquad + \gamma\big(T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big)\big)\big)\big).
\end{aligned}$$

This follows by

$$\begin{aligned}
&S^+\big(t, \omega, T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big) \\
&\qquad + \gamma\big(T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big)\big)\big) \\
&\quad = T(\tau, \theta_{t+\tau}\omega, \mu)(z^+), \\
&S^-\big(t, \omega, T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big) \\
&\qquad + \gamma\big(T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big)\big)\big) \\
&\quad = \mu\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big).
\end{aligned}$$

In particular, the last equation follows by (23). We can continue by

$$\begin{aligned}
\phi(\tau, \theta_t\omega, \mu)(z^+) &= S^-\big(t+\tau, \omega, T(t, \theta_t\omega, \gamma)T(\tau, \theta_{t+\tau}\omega, \mu)(z^+) \\
&\qquad + \gamma\big(T(t, \theta_t\omega, \gamma)\big(T(\tau, \theta_{t+\tau}\omega, \mu)(z^+)\big)\big)\big) \\
&= S^-\big(t+\tau, \omega, T(t+\tau, \theta_{t+\tau}\omega, \gamma)(z^+) \\
&\qquad + \gamma\big(T(t+\tau, \theta_{t+\tau}\omega, \gamma)(z^+)\big)\big) \\
&= \phi(t+\tau, \omega, \gamma)(z^+).
\end{aligned}$$

∎

Using the fact that the random graph transformation defines a cocycle we can look for a fixed point of the system (θ, ϕ).

THEOREM 3.6. *Suppose the cocycle defined by* (22) *has a stationary solution* $g^*(\omega, \cdot) \in G(\omega) \cap C_0^G(E^+(\omega), E^-(\omega))$ *so that* (3) *is fulfilled. For any* $x \in \mathbb{R}^d$ *the mapping,*

$$\omega \mapsto g^*(\omega, x)$$

is assumed to be measurable. Then we have a random invariant manifold $\{M(\omega)\}_{\omega \in \Omega}$ *such that* $M(\theta_t\omega) \supset S(t, \omega, M(\omega))$ *where* $M(\omega) = \{x^+ + g^*(\omega, x^+), x^+ \in E^+(\omega)\}$.

The proof is straightforward. Indeed, we have

$$
\begin{aligned}
&S(t,\omega,x^{+}+g^{*}(\omega,x^{+}))\\
&\qquad = S^{+}(t,\omega,x^{+}+g^{*}(\omega,x^{+})) + S^{-}(t,\omega,x^{+}+g^{*}(\omega,x^{+}))\\
&\qquad = S^{+}(t,\omega,x^{+}+g^{*}(\omega,x^{+})) + g^{*}\big(\theta_t\omega, S^{+}(t,\omega,x^{+}+g^{*}(\omega,x^{+}))\big)\\
&\qquad \in M^{+}(\theta_t\omega),
\end{aligned}
$$

for $x^{+}\in E^{+}(\omega)$. On the other hand we have $g^{*}(\omega,x) = \pi_{\omega}^{-}g^{*}(\omega,\pi_{\omega}^{+}x)$ by the definition of $\phi(t,\omega)$. By Lemma 3.2 the multifunction $M(\omega)$ is measurable. ∎

Particular assumptions will be formulated to ensure that this invariant manifold is unstable.

4. THE UNSTABLE MANIFOLD FOR A TWO-DIMENSIONAL RANDOM DIFFERENTIAL EQUATION

We now want to apply the random graph transformation to show that a particular two-dimensional differential equation has a random unstable invariant manifold. For this example we suppose that one of the equations of this system depends only on one variable, which allows us to formulate the conditions for existence of such a manifold in a weak form, that is in an *averaged* sense. For more general examples based on the method formulated in Section 3 we refer to Schmalfuss [16].

We consider the following two-dimensional differential equation,

$$
\begin{aligned}
\frac{du^{+}}{dt} &= a_{\theta_t\omega}^{+}u^{+} + b_{\theta_t\omega}^{+}(u^{+}), \qquad u^{+}(0) = x^{+}\in\mathbb{R},\\
\frac{du^{-}}{dt} &= a_{\theta_t\omega}^{-}u^{-} + b_{\theta_t\omega}^{-}(u^{+},u^{-}), \qquad u^{-}(0) = x^{-}\in\mathbb{R}.
\end{aligned}
\tag{24}
$$

We assume that the mappings,

$$
\omega\mapsto a_{\omega}^{\pm}, \qquad \Omega\times\mathbb{R}\ni(\omega,u^{+})\mapsto b_{\omega}^{+}(u^{+})\in\mathbb{R},
$$

$$
\Omega\times\mathbb{R}^{2}\ni(\omega,u)\mapsto b_{\omega}^{-}(u)\in\mathbb{R}
$$

are measurable. In addition, we assume that b^{+},b^{-} are differentiable at zero, that

$$
\begin{aligned}
&\mathbb{E}a^{+}>0, \qquad \mathbb{E}a^{-}<0, \qquad a^{+},a^{-}\in L^{1}(\Omega,\mathscr{F},\mathbb{P}),\\
&b_{\omega}^{+}(0) = Db_{\omega}^{+}(0) = 0,\\
&b_{\omega}^{-}(0) = Db_{\omega}^{-}(0) = 0, \quad \text{for } \omega\in\Omega,
\end{aligned}
\tag{25}
$$

and that $t \mapsto a^+_{\theta_t\omega}$, $t \mapsto a^-_{\theta_t\omega}$ are locally integrable for $\omega \in \Omega$. In addition, we suppose that the nonlinear part $b_\omega = (b^+_\omega, b^-_\omega)$ is globally Lipschitz continuous with Lipschitz constant L_ω, where $t \mapsto L_{\theta_t\omega}$ is locally integrable for $\omega \in \Omega$. These assumptions ensure that Eq. (24) has a unique solution, see Coddington and Levinson, Chap. 2, [6] which depends measurably on (t, ω, x^+, x^-).

We now consider the random dynamical system generated by Eq. (24). The operators $S(t, \omega, x^+, x^-)$ are given by the solution of Eq. (24) with initial condition (x^+, x^-) for a path ω at time t. The linearization of $S(t, \omega, x^+, x^-)$ at zero is the diagonal matrix,

$$S'(t, \omega, 0) = \begin{pmatrix} e^{\int_0^t a^+_{\theta_\tau\omega}\, d\tau} & 0 \\ 0 & e^{\int_0^t a^-_{\theta_\tau\omega}\, d\tau} \end{pmatrix}.$$

In this particular case we have two Oseledets spaces which are independent of ω where by (25) one of these spaces is the unstable linear space $E^+ = \operatorname{span}\{(1,0)\}$ and the other space is the stable linear space $E^- = \operatorname{span}\{(0,1)\}$. We now prove the existence of an unstable manifold if the expectation of the coefficients of (24) satisfies certain conditions.

THEOREM 4.1. *Assume for the coefficients of the random differential equation introduced in* (24) *that* $a^+, a^-, L \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, $\mathbb{E}a^+ > 0$, $\mathbb{E}a^- < 0$ *and*

$$\mathbb{E} \sup_{s \in [0,1]} \log^+ L_{\theta_s\omega} < \infty, \qquad \mathbb{E}\left(a^- - a^+ + \frac{5}{2}L\right) < 0, \qquad \mathbb{E}(-a^+ + L) < 0. \tag{26}$$

Then the random dynamical system (θ, S) *generated by* (24) *has a random unstable manifold represented by a Lipschitz continuous function.*

Proof. (i) We calculate some a priori estimates for the solution of (24). Because the coefficients of (24) are Lipschitz continuous the mapping $x \mapsto S(t, \omega, x)$ and $x^+ \mapsto u^+(t, \omega, x^+)$ have the same property. In particular, calculating the square of $u^+(t, \omega, x^+)$ by the chain rule gives

$$\|u^+(t, \omega, \cdot)\|_G \le \begin{cases} e^{\int_0^t a^+_{\theta_\tau\omega} + L_{\theta_\tau\omega}\, d\tau}, & t \ge 0 \\ e^{\int_0^{-t} -a^+_{\theta_{-\tau}\omega} + L_{\theta_{-\tau}\omega}\, d\tau}, & t \le 0. \end{cases} \tag{27}$$

The same relation is true if we estimate u^+ in the $\|\cdot\|_L$-norm. From now on we assume $\gamma \in C_0^{0,1} = C_0^{0,1}(E^+, E^-)$. We consider (24) with initial condition $(u^+(0), u^-(0)) = (x^+, \gamma(x^+))$. By the estimate,

$$2|b^-_\omega(u^+, u^-)|\,|u^-| \le 2L_\omega|u^+|\,|u^-| + 2L_\omega|u^-|^2 \le L_\omega|u^+|^2 + 3L_\omega|u^-|^2,$$

one finds for the second equation (24) in a similar manner to (27) for $t \geq 0$ that

$$\|S^{-}(t,\omega,\cdot+\gamma(\cdot))\|_G^2 \leq \exp\left(\int_0^t \left(2a^{-}_{\theta_\tau\omega}+3L_{\theta_\tau\omega}\right)d\tau\right)\|\gamma\|_G^2$$

$$+\int_0^t L_{\theta_\tau\omega}\|u^{+}(\tau,\omega,\cdot)\|_G^2$$

$$\cdot\exp\left(\int_\tau^t\left(2a^{-}_{\theta_s\omega}+3L_{\theta_s\omega}\right)ds\right)d\tau. \quad (28)$$

The same estimates remain true if we replace all $\|\cdot\|_G$-norms by $\|\cdot\|_L$-norms.

(ii) We introduce a cocycle on a function space. S^{+} is generated by u^{+}, hence

$$T(t,\omega,\gamma)(x^{+}) = u^{+}(-t,\omega,x^{+}),$$

which shows that $T(\cdot,\cdot,\gamma)$ is independent of γ. Indeed, this follows because the differential equation (24) is semicoupled. By Theorem 3.5,

$$\phi(t,\omega,\gamma) = S^{-}\big(t,\omega,T(t,\theta_t\omega)+\gamma\big(T(t,\theta_t\omega)\big)\big)$$

defines a cocycle on $C_0^{0,1}$.

We have by (27) if we replace ω by $\theta_t\omega$,

$$\big\|\gamma\big(T(t,\theta_t\omega)\big)\big\|_G \leq \exp\left(\int_0^t\left(-a^{+}_{\theta_\tau\omega}+L_{\theta_\tau\omega}\right)d\tau\right), \qquad t\geq 0.$$

Then it follows from (28) that

$$\|\phi(t,\omega,\gamma)\|_G^2 \leq \|\gamma\|_G^2 e^{\int_0^t 2a^{-}_{\theta_\tau\omega}-2a^{+}_{\theta_\tau\omega}+5L_{\theta_\tau\omega}\,d\tau}$$

$$+\int_0^t L_{\theta_\tau\omega}e^{\int_\tau^t 2a^{-}_{\theta_{\tau'}\omega}-2a^{+}_{\theta_{\tau'}\omega}+5L_{\theta_{\tau'}\omega}\,d\tau'}\,d\tau. \quad (29)$$

The same estimate is true if we replace $\|\cdot\|_G$ by $\|\cdot\|_L$.

(iii) We now check the assumptions of the fixed point Theorem 2.2. For the phase space we choose $G(\omega)\equiv C_0^{0,1}$. Let Ω' be a θ-invariant set of full measure such that

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\left(2a^{-}_{\theta_\tau\omega}-2a^{+}_{\theta_\tau\omega}+5L_{\theta_\tau\omega}\right)^{\pm}d\tau = \mathbb{E}(2a^{-}-2a^{+}+5L)^{\pm},$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\left(-a^{+}_{\theta_\tau\omega}+L_{\theta_\tau\omega}\right)d\tau = \mathbb{E}(-a^{+}+L)<0, \quad (30)$$

$$\lim_{t\to\pm\infty}\frac{\sup_{s\in[0,1]}\log^{-}L_{\theta_t\theta_s\omega}}{t} = 0.$$

The existence of such a θ-invariant set follows from the ergodic theorem and from (26). For the third property in (30) we have to use the fact that

$$\sup_{\tau\in[0,1]} \sup_{s\in[0,1]} \log^+ L_{\theta_\tau\theta_s\omega} \le \sup_{s\in[0,2]} \log^+ L_{\theta_s\omega},$$

and that the expectation of the right-hand side is finite, which follows immediately from the first condition in (26). A consequence of the first and third property of (30) is that

$$\begin{aligned}\lim_{t\to\pm\infty} \frac{\int_0^\tau |2a^-_{\theta_t\theta_s\omega} - 2a^+_{\theta_t\theta_s\omega} + 5L_{\theta_t\theta_s\omega}|\,ds}{t} &= 0,\\ \lim_{t\to\pm\infty} \frac{\sup_{s\in[0,\tau]} \log^+ L_{\theta_s\theta_t\omega}}{t} &= 0,\end{aligned} \tag{31}$$

for any $\omega \in \Omega'$, $\tau > 0$.

Let $\mathscr{G}_{\Omega'}$ be the set of mappings defined on Ω' with values in $C_0^{0,1}$ such that for $g \in \mathscr{G}_{\Omega'}$,

$$w \mapsto \|g(\omega)\|_G$$

is tempered. In addition, we suppose that for fixed $x \in \mathbb{R}^d$,

$$\Omega' \ni \omega \mapsto g(\omega, x)$$

is measurable. We note that in contrast to Remark 2.1 we included in the definition of $\mathscr{G}_{\Omega'}$ the stronger temperedness w.r.t. continuous time. Then $\|g_1 - g_2\|_G$ fulfills the assumption of Theorem 2.2 for $g_1, g_2 \in \mathscr{G}_{\Omega'}$. To ensure the invariance we have to check that $\omega \mapsto \|\phi(\tau, \omega, g(\omega))\|_G$ is also tempered for any $\tau > 0$, $g \in \mathscr{G}_{\Omega'}$. On account of (29) it is sufficient to show that

$$\lim_{t\to\pm\infty} \left(\frac{\log\left(\max\left(\|g(\theta_t\omega)\|_G^2, 1\right)\right)}{|t|} + \frac{1}{|t|}\int_0^\tau |2a^-_{\theta_t\theta_s\omega} - 2a^+_{\theta_t\theta_s\omega} + 5L_{\theta_t\theta_s\omega}|\,ds \right) = 0,$$

$$\lim_{t\to\pm\infty} \frac{\log^+ \int_0^\tau L_{\theta_s\theta_t\omega} \exp\left(\int_s^\tau \left(2a^-_{\theta_t\theta_{s'}\omega} - 2a^+_{\theta_t\theta_{s'}\omega} + 5L_{\theta_t\theta_{s'}\omega}\right) ds'\right) ds}{|t|} = 0.$$

The first relation follows straightforwardly from $g \in \mathscr{G}_{\Omega'}$ and the first relation in (31). The expression in the last formula can be estimated by

$$\frac{1}{|t|}\log^+\int_0^{\tau+1}\exp\left(\sup_{s\in[0,\tau]}\log^+ L_{\theta_s\theta_t\omega}+\int_0^\tau\left|2a^-_{\theta_t\theta_{s'}\omega}-2a^+_{\theta_t\theta_{s'}\omega}+5L_{\theta_t\theta_{s'}\omega}\right|ds'\right)ds$$
$$\leq \frac{1}{|t|}\left(\sup_{s\in[0,\tau]}\log^+ L_{\theta_s\theta_t\omega}\right.$$
$$\left.+\int_0^\tau\left|2a^-_{\theta_t\theta_{s'}\omega}-2a^+_{\theta_t\theta_{s'}\omega}+5L_{\theta_t\theta_{s'}\omega}\right|ds'+\log(\tau+1)\right),$$

which proves the asserted property.

We now show that if $(\phi(t,\theta_{-t}\,\omega, g(\theta_{-t}\,\omega)))$ is a Cauchy sequence, then this limit is contained in $\mathscr{G}_{\Omega'}$. Using the abbreviation $A_\omega = 2a^-_\omega - 2a^+_\omega + 5L_\omega$ we find by (29) replacing ω by $\theta_{-t}\omega$ that any limit point of $(\phi(t(\theta_{-t}\,\omega, g(\theta_{-t}\,\omega)))$ for $t\to\infty$ is contained in a C_0^G-ball with center zero and radius,

$$\int_{-\infty}^0 L_{\theta_\tau\omega}e^{\int_\tau^0 A_{\theta_{\tau'}\omega}\,d\tau'}\,d\tau.$$

This term is finite for $\omega\in\Omega'$ by (30). Thus we have to show the temperedness of this term. Sufficient for this temperedness is that

$$\lim_{t\to\pm\infty} e^{-c|t|}\int_{-\infty}^0 e^{\log^+ L_{\theta_{t+\tau}\omega}+\int_{t+\tau}^t A_{\theta_s\omega}\,ds}\,d\tau = 0,$$

for any $c>0$. We consider only the case $t\to-\infty$, as in the case $t\to\infty$ we can establish the convergence to zero similarly. We can write this formula as

$$e^{ct}\int_{-\infty}^0 e^{\log^+ L_{\theta_{t+\tau}\omega}+\int_{t+\tau}^t(A_{\theta_s\omega}-\mathbb{E}A)\,ds-\tau\mathbb{E}A}\,d\tau$$
$$= e^{ct}\int_{-\infty}^0 e^{\log^+ L_{\theta_{t+\tau}\omega}+\int_{t+\tau}^0(A_{\theta_s\omega}-\mathbb{E}A)\,ds-\int_t^0(A_{\theta_s\omega}-\mathbb{E}A)\,ds-\tau\mathbb{E}A}\,d\tau. \quad (32)$$

The assumption of the theorem gives for any $\varepsilon>0$, $\omega\in\Omega'$ a $t(\omega,\varepsilon)<0$ such that for $t<t(\omega,\varepsilon)$,

$$\log^+ L_{\theta_t\omega}\leq\varepsilon|t|, \qquad \int_t^0\left(A_{\theta_s\omega}-\mathbb{E}A\right)ds\leq\varepsilon|t|.$$

We now choose ε so that $0 < 4\varepsilon < \min(c/2, -\mathbb{E}A)$. Thus the integral in (32) is bounded by

$$e^{c(t/2)} \int_{-\infty}^{0} e^{\varepsilon\tau}\, d\tau,$$

for sufficiently large $|t|$, which gives the convergence of the last formulae to zero for $t \to -\infty$.

Applying (29) for the $C_0^{0,1}$-norm one can find in a similar manner that for any $g \in \mathscr{G}_{\Omega'}$ also the sequence $(\phi(t, \theta_{-t}\omega, g(\theta_{-t}\omega)))$ is contained in a $C_0^{0,1}$-ball with center 0 for any $\omega \in \Omega'$. By Lemma 33.4 this limit is contained in $C_0^{0,1}$.

We now check the contraction condition. The expression,

$$\Delta(t) := u^-(t, \omega, \gamma_1(x^+)) - u^-(t, \omega, \gamma_2(x^+))$$

satisfies

$$\frac{d\Delta(t)}{dt} = a^-_{\theta_t\omega}\Delta(t) + B(t, \omega)\Delta(t),$$

where $B(t, \omega)$ is given by the expression,

$$\frac{b^-_{\theta_t\omega}(u^+(t, \omega, x^+), u^-(t, \omega, \gamma_1(x^+))) - b^-_{\theta_t\omega}(u^+(t, \omega, x^+), u^-(t, \omega, \gamma_2(x^+)))}{\Delta(t)},$$

(if $\Delta(t) \neq 0$). We have the initial condition $\gamma_1(x^+) - \gamma_2(x^+)$ for $\gamma_i \in C_0^{0,1}$, $i = 1, 2$. The Lipschitz continuity of the coefficient b yields

$$\|\Delta(t)\|_G \le \exp\left(\int_0^t a^-_{\theta_\tau\omega} + L_{\theta_\tau\omega}\, d\tau\right)\|\gamma_1 - \gamma_2\|_G.$$

We also have

$$\left\|\gamma_1(T(t, \theta_t\omega, \cdot)) - \gamma_2(T(t, \theta_t\omega, \cdot))\right\|_G \le \|\gamma_1 - \gamma_2\|_G \|T(\theta_t\omega, \cdot)\|_G,$$

and by (27),

$$\|T(t, \theta_t\omega, \cdot)\|_G \le \exp\left(\int_0^t \left(-a^+_{\theta_\tau\omega} + L_{\theta_\tau\omega}\right) d\tau\right),$$

such that the contraction condition of Theorem 2.2 can be represented by

$$k(\omega) = \int_0^1 \left(a^-_{\theta_\tau\omega} - a^+_{\theta_\tau\omega} + \frac{5}{2}L_{\theta_\tau\omega}\right) d\tau, \qquad K = \mathbb{E}\left(a^- - a^+ - \frac{5}{2}L\right) < 0.$$

The application of Theorem 2.2 and Remark 3.3 ensures the existence of the fixed point g^* in $C_0^{0,1}$ which is tempered w.r.t. C_0^G. The mapping,

$$\mathbb{T} \times \Omega' \times \mathbb{R}^d \ni (t, \omega, x) \mapsto S(t, \omega, x) \in \mathbb{R}^2$$

is measurable. Thus $\omega \mapsto \phi(t, \omega, g(\omega))(x^+)$ is also measurable for any $g \in \mathscr{G}_{\Omega'}$, $x^+ \in \mathbb{R}$ which ensures by (11) that for fixed x^+ the mapping,

$$\omega \mapsto g^*(\omega, x^+)$$

is measurable such that by Lemma 3.2 g^* is the graph of an invariant manifold.

By the third inequality in (26) the solution $u^+(t, \omega, x^+)$ tends to zero exponentially fast for $t \to -\infty$ and $\omega \in \Omega'$. On the other hand we have

$$|g^*(\theta_t \omega, u^+(t, \omega, x^+))| \leq \|g^*(\theta_t \omega)\|_G |u^+(t, \omega, x^+)|.$$

By the temperedness of $\|g^*(\omega)\|_G$ we have the exponential convergence of the right-hand side for $t \to -\infty$ and for any x^+ which shows that g^* is the graph of the unstable manifold. ∎

ACKNOWLEDGMENTS

This article was partially written during my appointment at the Institut für Dynamische Systeme, Universität Bremen. I thank Ludwig Arnold and Hannes Keller for valuable suggestions. The second part was written in the research group of Sri Namachchivaya at the University of Illinois at Urbana Champaign supported by a travel grant of the *Deutsche Forschungsgemeinschaft.*

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